

UNIFORM HARBOURNE-HUNEKE BOUNDS VIA FLAT EXTENSIONS

ROBERT M. WALKER

ABSTRACT. Over an arbitrary field \mathbb{F} , Harbourne and Huneke ([10]) conjectured in 2011 that

$$I^{(N(r-1)+1)} \subseteq I^r$$

for all $r > 0$ and all homogeneous ideals I in $S = \mathbb{F}[\mathbb{P}^N] = \mathbb{F}[x_0, \dots, x_N]$. The conjecture has been disproven for select values of $N \geq 2$: first by Dumnicki, Szemberg, and Tutaj-Gasińska in characteristic zero ([6]), and then by Harbourne and Secoreanu in positive characteristic ([11]). However, the ideal containments above do hold when, for instance, I is a monomial ideal in S .

This manuscript is in part a sequel to preprint ([20]), and presents criteria for containments of type $I^{(N(r-1)+1)} \subseteq I^r$ for all $r > 0$ and certain classes of ideals I in a prodigious class of normal rings (e.g., coordinate rings of simplicial toric varieties). Of particular interest is a result for monomial primes in tensor products of affine semigroup rings. Indeed, we explain how to give effective multipliers N in several cases including: the D -th Veronese subring of any polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ ($n \geq 1$); and the extension ring $\mathbb{F}[x_1, \dots, x_n, z]/(z^D - x_1 \cdots x_n)$ of $\mathbb{F}[x_1, \dots, x_n]$.

CONTENTS

1. Introduction and Conventions for the Paper	1
2. Symbolic Powers and Faithful Flatness; The Proof of Lemma 1.1	3
3. The Proof of Theorem 1.1: New Examples from Old	5
4. Proving Theorem 1.2 in a Refined Form	10
5. Lingering Questions related to Theorem 1.1	15
References	16

1. INTRODUCTION AND CONVENTIONS FOR THE PAPER

Over an arbitrary field \mathbb{F} , let $S = \mathbb{F}[\mathbb{P}^N] = \mathbb{F}[x_0, x_1, \dots, x_N]$ be the standard \mathbb{N} -graded polynomial algebra. The Ein-Lazarsfeld-Smith Theorem ([7, 15]), as extended by Hochster and Huneke, implies that the symbolic power $I^{(Nr)} \subseteq I^r$ for all graded ideals $0 \subsetneq I \subsetneq S$ and all integers $r > 0$. Using graded ideals of *star configurations* in \mathbb{P}^N , Bocci and Harbourne ([4]) showed that in securing these containments one cannot replace N by some integer $0 < C < N$. As part of an active area of research into symbolic powers in S , **ideal containment problems** concern the determination of when select families of ideal containments hold for a graded ideal $0 \subsetneq I \subsetneq S$, e.g., those of the type

2010 *Mathematics Subject Classification*: 13H10, 14C20, 14M25.

Keywords: symbolic powers, divisor class group, rational singularity, toric variety.

$I^{(m)} \subseteq I^r$. Of particular interest in this paper, in 2011 Harbourne and Huneke ([10], Conjecture 4.1.1) proposed dropping the symbolic power from Nr down to the **Harbourne-Huneke bound** $Nr - (N - 1) = N(r - 1) + 1$ when $N \geq 2$ (that is, dropping it as much as possible without a known counterexample at the time); thus they conjectured that for any graded ideal $0 \subsetneq I \subsetneq S$

$$I^{(N(r-1)+1)} \subseteq I^r \text{ for all } r > 0 \text{ and all } N \geq 2. \quad (1.0.1)$$

This conjecture started from the case $N = r = 2$: $I^{(4)} \subseteq I^2$ holds for all graded ideals in $\mathbb{F}[\mathbb{P}^2]$ by the Ein-Lazarsfeld-Smith Theorem; Huneke asked whether an improvement $I^{(3)} \subseteq I^2$ holds for any radical ideal I defining a finite set of points in \mathbb{P}^2 .

Along with ([10], subsection 4.1), a recent survey by Szemberg and Szpond ([19], Theorem 3.8) summarizes scenarios in which a particular containment for a given r is known and even classes of ideals for which this conjecture has been confirmed (e.g., over arbitrary fields, it holds for ideals of star configurations and for monomial ideals). However, it has been known since the counterexamples of Dumnicki, Szemberg, and Tutaj-Gasińska in characteristic zero ([6], 2013) that the containment $I^{(3)} \subseteq I^2$ can fail for the defining ideal I of a point configuration in \mathbb{P}^2 , whence (1.0.1) above is not kismet. It has been known since the Harbourne-Seceleanu counterexamples in odd positive characteristic ([11]) that (1.0.1) can fail for pairs $(N, r) \neq (2, 2)$ and ideals I defining a point configuration in \mathbb{P}^N . The recent note by Akeseh ([1]) shows how to use finite, flat morphisms $\varphi^\# : \mathbb{P}^N \rightarrow \mathbb{P}^N$ to swiftly cook up many new counterexamples to (1.0.1) from previous ones.

Lately, there has been better sustained success in showing that a containment in (1.0.1) fails—and perhaps, more fervor. However, we want to revisit the fact that in arbitrary characteristic (1.0.1) holds for all monomial ideals in S . In particular, our investigation of Harbourne-Huneke bounds *improves upon* the fact that $P^{(N(r-1)+1)} \subseteq P^{(r)} = P^r$ for all $r > 0$ and for all monomial prime ideals P in S (i.e., monomial ideals generated by subsets of the variables x_0, \dots, x_N). Indeed, $P^{(r)} = P^r$ for all r whenever P is a complete intersection (CI) ideal in S , and since monomial primes are CI ideals, the containments follow quickly upon noticing that for all $r > 0$, $N(r-1)+1 \geq (r-1)+1 = r$.

The goal of this paper is to show that a variant of (1.0.1) holds for several familiar classes of ideals (e.g., combinatorial ideals such as monomial primes) in certain non-regular rings—even though it already fails for a large class of ideals defining point configurations in \mathbb{P}^N , and hence can fail for arbitrary graded ideals in $\mathbb{F}[\mathbb{P}^N]$. More precisely, we work in the setting of rational surface singularities and higher-dimensional normal toric rings. First, we demonstrate how one can strengthen Lemmas 1.1 and 2.3 of ([20]) to a version involving a Harbourne-Huneke bound:

Lemma 1.1. *Let R be a Noetherian normal domain of positive Krull dimension whose global divisor class group $\text{Cl}(R) := \text{Cl}(\text{Spec}(R))$ is annihilated by an integer $D > 0$. Then*

$$\mathfrak{q}^{(D(r-1)+s)} = (\mathfrak{q}^{(D)})^{r-1} \mathfrak{q}^{(s)}, \text{ and } \mathfrak{q}^{(D(r-1)+1)} \subseteq \mathfrak{q}^r$$

for all ideals $\mathfrak{q} \subseteq R$ of pure height one, all $r > 0$, and all $0 \leq s < D$.

In particular, when $D = 2$ works, we have $\mathfrak{q}^{(3)} \subseteq \mathfrak{q}^2$ for all ideals $\mathfrak{q} \subseteq R$ of pure height one. When the domain R in this lemma is two-dimensional, $P^{(r)} = P^r$ when the ideal P is zero or maximal, and so we infer that $P^{(D(r-1)+1)} \subseteq P^r$ for all prime ideals P in R and all $r > 0$, and that $P^{(3)} \subseteq P^2$ for all primes when $D = 2$ works. As discussed in ([20]), the above lemma already applies to any two-dimensional, local rational singularity (Lipman [16]) and the coordinate rings of simplicial toric varieties; see Theorem (4.2) below. The intro to ([20]) gives Lipman's definition of two-dimensional, normal local rational singularities; section 3 therein gives remarks on class groups, both for these

singularities and for toric varieties. We prove a result for Veronese rings (Theorem (4.3) below) from which one can infer that the ideal containment in the lemma can be **tight**.

However, it is the result to follow that inspires the chosen title for this paper. It allows us to give first examples of the Harbourne-Huneke bound for all monomial primes in certain normal algebras of dimension three or higher, subalgebras of a Laurent polynomial ring that are generated by monomials. These domains are the coordinate rings of normal affine toric varieties, called toric rings or affine semigroup rings. In this setting, we adduce a result (Proposition (2.1)) on ideal containment preservation along faithfully flat ring extensions, as part of deducing the following

Theorem 1.1. *Let R_1, \dots, R_n be normal affine semigroup rings over a field \mathbb{F} . For each $1 \leq i \leq n$, suppose there is an integer $D_i > 0$ such that $P^{(D_i(r-1)+1)} \subseteq P^r$ for all $r > 0$ and all monomial primes $P \subseteq R_i$. Set $D := \max\{D_1, \dots, D_n\}$. Then $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$ and any monomial prime Q in the normal affine semigroup ring $R = R_1 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n$.*

All normal toric rings of dimension at most two have finite cyclic class group, and thus satisfy the hypotheses on the R_i factors in the theorem; aside from these cases, the factors R_i may be taken from the following classes of rings (including those of Krull dimension three or higher):

Theorem 1.2. *Let $S = \mathbb{F}[x_1, \dots, x_n]$ ($n \geq 1$) be a polynomial ring over an arbitrary field \mathbb{F} and consider the module-finite extensions of normal toric rings $V_D \subseteq S \subseteq H_D$, where*

- (1) $V_D \subseteq S$ is the D -th Veronese subring with its standard \mathbb{N} -grading, and
- (2) $H_D = \mathbb{F}[z, x_1, \dots, x_n]/(z^D - x_1 \dots x_n)$ is a hypersurface ring.

Then $P^{(D(r-1)+1)} \subseteq P^r$ for all $r > 0$ and all monomial prime ideals P in each ring.

The paper is organized as follows. In Section (2), we prove Lemma (1.1) and Proposition (2.1). Section (3) reviews most of the relevant toric algebraic geometry, culminating in a proof of Theorem (1.1). In Section (4), we prove Theorem (1.2), along with clarifying that theorem's connection with divisor class groups. Section (5) is succinct, closing the paper with some lingering questions.

Conventions: All our rings are Noetherian and commutative with identity. From section (3), our rings will be *affine* \mathbb{F} -algebras, that is, of finite type over a fixed field \mathbb{F} of arbitrary characteristic. By *algebraic variety*, we will mean an integral scheme of finite type over the field \mathbb{F} .

Acknowledgements: I thank my thesis adviser, Karen E. Smith, for encouraging me to write this manuscript, and for several fruitful discussions along the way. I also thank Daniel Hernández, Jack Jeffries, Luis Núñez-Betancourt, and Felipe Pérez for each critiquing a draft of the paper. I was supported by a NSF GRF under Grant Number PGF-031543, and this work was partially supported by the NSF RTG grant 0943832.

2. SYMBOLIC POWERS AND FAITHFUL FLATNESS; THE PROOF OF LEMMA 1.1

Symbolic Powers and Faithful Flatness: If I is any proper ideal in a nonzero Noetherian ring R , and $\text{Ass}_R(R/I)$ is the set of associated primes of I , we define its a -**th** ($a \in \mathbb{Z}_{>0}$) **symbolic power** ideal $I^{(a)}$ by the rule:

$$f \in I^{(a)} \iff sf \in I^a \text{ for some } s \in S := R - \left(\bigcup_{P \in \text{Ass}_R(R/I)} P \right) = \bigcap_{P \in \text{Ass}_R(R/I)} (R - P).$$

Per (Atiyah-Macdonald [3], Proposition 4.9), $I^{(1)} = I$ for any proper ideal $I \subseteq R$. In general, $I^{(a)} \supseteq I^a$ for all $a > 1$.

Consider a flat map $\phi: A \rightarrow B$ of Noetherian rings. In what follows, the ideal $JB := \langle \phi(J) \rangle B$ for any ideal J in A , and $J^r B = (JB)^r$ for all $r \geq 0$ since the two ideals share a generating set. For any A -module E , the proof of Theorem 23.2 (ii) in Matsumura ([17]) shows that

$$\text{Ass}_B(E \otimes_A B) = \bigcup_{P \in \text{Ass}_A(E)} \text{Ass}_B(B/PB). \quad (2.0.2)$$

We define a set $\mathcal{I}(A) = \{\text{proper ideals } I \subseteq A: \text{Ass}_B(B/IB) = \{PB: P \in \text{Ass}_A(A/I)\}\}$. Setting $E = A/I$ in (2.0.2), we observe that $I \in \mathcal{I}(A)$ if and only if the extended ideal PB is prime for all $P \in \text{Ass}_A(A/I)$.

Proposition 2.1. *Suppose $\phi: A \rightarrow B$ is a faithfully flat map of Noetherian rings. Then for each $I \in \mathcal{I}(A)$ and all integer pairs $(N, r) \in (\mathbb{Z}_{\geq 0})^2$, we have*

$$I^{(N)}B = (IB)^{(N)}, \quad (2.0.3)$$

and $I^{(N)} \subseteq I^r$ if and only if $(IB)^{(N)} = I^{(N)}B \subseteq I^r B = (IB)^r$.

Proof. First, $I^{(N)}B \subseteq (IB)^{(N)}$: indeed, if $f \in I^{(N)}$, then $sf \in I^N$ for some $s \in A$ such that

$$s \notin \bigcup_{P \in \text{Ass}_A(A/I)} P \stackrel{(\star)}{=} \bigcup_{P \in \text{Ass}_A(A/I)} (PB \cap A) = \left(\bigcup_{P \in \text{Ass}_A(A/I)} PB \right) \cap A$$

where (\star) holds by faithful flatness; it follows that $s \notin \bigcup_{P \in \text{Ass}_A(A/I)} PB = \bigcup_{Q \in \text{Ass}_B(B/IB)} Q$, where equality holds since $I \in \mathcal{I}(A)$ by hypothesis. We thus conclude that $f \in (IB)^{(N)}$.

By definition, $(IB)^{(N)}B_W = (IB)^NB_W = I^NB_W$ since all three ideals contract to $(IB)^{(N)}$, where B_W is the localization of B at the multiplicative system

$$W = B - \left(\bigcup_{Q \in \text{Ass}_B(B/IB)} Q \right) = B - \left(\bigcup_{P \in \text{Ass}_A(A/I)} PB \right).$$

Notice that since $I^{(N)}B \subseteq (IB)^{(N)}$, the right-hand containment holds in

$$I^NB_W \subseteq I^{(N)}B_W = (I^{(N)}B)B_W \subseteq (IB)^{(N)}B_W = I^NB_W.$$

Thus $I^{(N)}B$ and $(IB)^{(N)}$ localize to the same ideal I^NB_W ; contracting back to B , we conclude that (2.0.3) holds for all $N \geq 0$. Finally, (2.0.3) gives both implications of the second part of the proposition, adducing faithful flatness once more to contract an ideal containment to A . \square

We adapt Proposition (2.1) in the next section (cf., Proposition (3.4)) to prove Theorem (3.1).

The Proof of Lemma (1.1). For the remainder of this section, R will denote a Noetherian normal domain, and \mathcal{P} is the set of height-one primes in R . As noted in Matsumura's chapter on Krull rings ([17], Corollary of Theorem 12.3), when $f \in R$ is a nonzero nonunit, and ν_P is the discrete valuation on the DVR R_P (for $P \in \mathcal{P}$), we have a unique primary decomposition

$$(f)R = \bigcap_{P \in \mathcal{P}} P^{(N_P)}, \text{ where } N_P := \nu_P(f) = 0 \text{ for all but finitely many } P.$$

We define the **Weil divisor of f** to be $\text{div}(f) := \sum_{P \in \mathcal{P}} N_P \cdot P$. Additionally, we define the *trivial* effective Weil divisor $\text{div}(\langle 1 \rangle R) = \text{div}(R) = [R] := 0$ of the unit ideal to have identically zero \mathbb{Z} -coefficients. More generally, any ideal of **pure height one** (i.e., all associated primes have height one) has a unique primary decomposition as above, so we can define a Weil divisor for any such ideal (see below). The **divisor class group** $\text{Cl}(R) = \text{Cl}(\text{Spec}(R))$ of R is the free abelian group on \mathcal{P} modulo relations of the form

$$a_1 P_1 + \dots + a_r P_r = 0$$

whenever the ideal $P_1^{(a_1)} \cap \dots \cap P_r^{(a_r)}$ is principal. According to Proposition 6.2 in ([12], Ch.II, §6), $\text{Cl}(R) = 0$ if and only if R is a UFD (that is, every height-one prime ideal is principal). Note that $P^{(a)} = P^a$ for all height-one primes P in a UFD.

Proposition 2.2 (cf., Prop 2.2 of ([20])). *Let R be a Noetherian normal domain of positive Krull dimension, and \mathfrak{q} any ideal of pure height one with associated primes P_1, \dots, P_c . Then:*

- (a) *There exist positive integers b_1, \dots, b_c , uniquely determined by \mathfrak{q} , such that the symbolic power $\mathfrak{q}^{(E)} = P_1^{(Eb_1)} \cap \dots \cap P_c^{(Eb_c)}$ for all $E \geq 0$.*
- (b) *If either (1) $D \cdot \text{Cl}(R) = 0$, or (2) the class $[\mathfrak{q}] \in \text{Cl}(R)$ has finite order D , then for all integers $r \geq 0$, $\mathfrak{q}^{(Dr)} = (\mathfrak{q}^{(D)})^r$ is principal and $\mathfrak{q}^{(Dr)} \subseteq \mathfrak{q}^r$.*

Per part (a) of this proposition, we may define Weil divisors

$$\text{div}[\mathfrak{q}] := b_1 \cdot P_1 + \dots + b_c \cdot P_c, \quad \text{div}[\mathfrak{q}^{(E)}] := E \cdot \text{div}[\mathfrak{q}] = Eb_1 \cdot P_1 + \dots + Eb_c \cdot P_c \text{ for each } E > 0.$$

In particular, $\text{div}[\mathfrak{q}^{(A+B)}] = \text{div}[\mathfrak{q}^{(A)}] + \text{div}[\mathfrak{q}^{(B)}]$ for all pairs of nonnegative integers A, B .

Proof of Lemma (1.1). Our proof of the first claim replaces $r - 1$ with $r \geq 0$. Per Proposition (2.2)(b), suppose $\mathfrak{q}^{(Dr)} = (\mathfrak{q}^{(D)})^r = (f^r)$ is principal for all $r \geq 0$ and some nonzero $f \in R$. Now set $I = \mathfrak{q}^{(s)}$. Following the first proof in Hochster's notes ([14]), we have a short exact sequence

$$0 \rightarrow \frac{(f^r)R}{(f^r)I} \rightarrow \frac{R}{(f^r)I} \rightarrow \frac{R}{(f^r)R} \rightarrow 0$$

and $\frac{(f^r)R}{(f^r)I} \cong R/I$ as R -modules via the R -linear map $\phi: R \rightarrow \frac{(f^r)R}{(f^r)I}$ with $\phi(g) = \overline{gf^r}$. Thus per our exact sequence, $\emptyset \neq \text{Ass}_R(R/(f^r)I) \subseteq \text{Ass}_R(R/I) \cup \text{Ass}_R(R/(f^r)R)$ (cf., Thm 6.3 of Matsumura [17]) and so $\text{Ass}_R(R/(f^r)I)$ contains only height one primes since the latter two sets do. Finally, comparing Weil divisors of pure height one ideals

$$\boxed{\text{div}[(f^r)I = (\mathfrak{q}^{(D)})^r \mathfrak{q}^{(s)}] \stackrel{(*)}{=} \text{div}[(f^r)R] + \text{div}[I] = \text{div}[\mathfrak{q}^{(Dr)}] + \text{div}[\mathfrak{q}^{(s)}] = \text{div}[\mathfrak{q}^{(Dr+s)}].}$$

As Hochster notes, one can check identity $(*)$ after first localizing at each height one prime Q ; in this case, the identity is obvious in a DVR. Per $(*)$, the two pure height one ideals $\mathfrak{q}^{(Dr+s)}, (\mathfrak{q}^{(D)})^r \mathfrak{q}^{(s)}$ have the exact same primary decomposition and hence are equal. To conclude: since $\mathfrak{q}^{(D)} \subseteq \mathfrak{q}^{(1)} = \mathfrak{q}$, setting $s = 1$ yields $\mathfrak{q}^{(D(r-1)+1)} = (\mathfrak{q}^{(D)})^{r-1} \mathfrak{q}^{(1)} \subseteq \mathfrak{q}^{r-1+1} = \mathfrak{q}^r$. \square

3. THE PROOF OF THEOREM 1.1: NEW EXAMPLES FROM OLD

Toric Preliminaries: We cover enough commutative algebra and convex geometry under the hood of toric geometry to set up a proof of the theorem. We work over an **arbitrary** field \mathbb{F} . For simplicity, we conduct our review with the standard lattice \mathbb{Z}^n in \mathbb{R}^n . We follow: sections 1.2-3, 3.1-2, 4.1-2 of Cox-Little-Schenck ([5]); or alternatively, Chapters 1 and 3 of Fulton ([8]).

Any normal affine toric n -fold can be obtained from a strongly convex, rational polyhedral cone σ in \mathbb{R}^n ; we generally abbreviate this, saying only that σ is (SCR). First, a **polyhedral cone** in \mathbb{R}^n is a closed, convex set

$$\sigma = \text{Cone}(G) = \left\{ \sum_{v \in G} a_v \cdot v : \text{each } a_v \in \mathbb{R}_{\geq 0} \right\} \subseteq \mathbb{R}^n,$$

generated by a finite set G (possibly empty) of nonzero vectors. The strong convexity (SC) condition is simply that in addition to being convex, σ contains no line through the origin. The rationality (R) condition is simply that $G \subseteq \mathbb{Z}^n$. Note that the cone has **dimension** $\dim \sigma := \dim(\mathbb{R}\text{-linear span of } G) \leq n$; if $\dim \sigma = n$, then σ is **full** and we say σ is (SCRF). In particular, the **zero(-dimensional) cone** $\{0\} = \text{Cone}(\emptyset)$ consists only of the origin. Since $\sigma = \text{Cone}(G) \subseteq \mathbb{R}^n$ is rational, its **dual** polyhedral cone

$$\sigma^\vee := \{w \in \mathbb{R}^n : \langle w, v \rangle \geq 0 \text{ for all } v \in \sigma\} = \{w \in \mathbb{R}^n : \langle w, v \rangle \geq 0 \text{ for all } v \in G\}$$

is also rational with respect to \mathbb{Z}^n , where $\langle \cdot, \cdot \rangle$ is dot product; σ^\vee in turn yields

- (1) A finitely-generated semigroup under addition

$$(S_\sigma, +) := \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0}[m_1, \dots, m_r] = \left\{ \sum_{i=1}^r a_i m_i : \text{each } a_i \in \mathbb{Z}_{\geq 0} \right\}$$

for some finite list of generators $m_1, \dots, m_r \in \mathbb{Z}^n$. **Note:** strong convexity of σ is equivalent to σ^\vee being full, and in this case, $S_\sigma = \mathbb{Z}_{\geq 0}[\mathcal{H}_\sigma]$ for a unique generating set \mathcal{H}_σ (the **Hilbert basis** of S_σ) that is minimal with respect to inclusion among all generating sets of S_σ . In practice, we use the Polyhedra package in Macaulay2 ([9]) to compute Hilbert bases and to justify our \mathbb{F} -algebra presentations of toric rings; see Section (4).

- (2) A normal domain of finite type over \mathbb{F} : namely, the semigroup ring $R = \mathbb{F}[S_\sigma]$ generated as an algebra by the characters χ^m with $m \in S_\sigma$. If $\mathcal{H}_\sigma = \{m_1, \dots, m_r\}$, then as an algebra $R = \mathbb{F}[\chi^{m_1}, \dots, \chi^{m_r}]$. In computations, we regard $\chi^a = t_1^{a_1} \cdots t_n^{a_n}$ ($a = (a_1, \dots, a_n)$) as a Laurent monomial in n variables, whence R is a subring of the domain $\mathbb{F}[S_{\{0\}}] = \mathbb{F}[\mathbb{Z}^n] \cong \mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials in n variables over \mathbb{F} . Any domain so obtained is a **toric ring** (or normal affine semigroup algebra) over \mathbb{F} , and can be graded by \mathbb{Z}^n or S_σ ($\deg(\chi^{m_i}) = m_i$) so that $R_0 = \mathbb{F}$ ($0 \in \mathbb{Z}^n$) and there is a unique homogeneous maximal ideal. It follows from work of Hochster ([13]) that toric rings are Cohen-Macaulay.
- (3) A normal toric n -fold, assuming σ is (SCR): define a normal affine variety $U_\sigma = \text{Spec}(R)$; by *variety*, we mean an integral scheme of finite type over a field. Strong convexity ensures that the n -torus $(\mathbb{F}^\times)^n = \text{Spec}(\mathbb{F}[\mathbb{Z}^n])$ embeds as a dense open set in U_σ , whence

$$\mathbb{F}(U_\sigma) = \text{frac}(R) = \text{frac}(\mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]) = \mathbb{F}(t_1, \dots, t_n) \implies \dim(U_\sigma) = \dim(R) = n.$$

An (SCR) cone $\sigma \subseteq \mathbb{R}^n$ is **simplicial** if $\sigma = \text{Cone}(G)$ for some $G \subseteq \mathbb{Z}^n$ forming part of a \mathbb{R} -basis for \mathbb{R}^n , and we call the corresponding toric ring and toric variety **simplicial**. Moreover, we may assume $G = \{v_1, \dots, v_{\dim(\sigma)}\}$ consists of **primitive** vectors, i.e., the coordinates of each v_i have no common prime integer factor. In the simplicial case, σ has $\binom{\dim(\sigma)}{\ell}$ faces of dimension ℓ . Roughly, a face is a polyhedral subcone of σ obtained by intersecting σ with any appropriately-chosen supporting hyperplane in \mathbb{R}^n . We count σ as a face of itself. If σ is (SCR), then so are all of its (finitely-many) faces.

3.0.1. *Monomial Primes and Faces.* We adjust a passage from Fulton ([8], p.53) to a form that streamlines some computations. Take a face $\tau = \text{Cone}(T)$ of an (SCR) cone $\sigma = \text{Cone}(S) \subseteq \mathbb{R}^n$, where $T \subseteq S$, and where S consists of the primitive ray generators. **By convention, we count σ as a face of itself.** To clarify, a **rational ray** ρ is an (SCR) cone of dimension 1. The semigroup $\rho \cap \mathbb{Z}^n$ has Hilbert basis consisting of a single primitive vector u_ρ , the **(primitive) ray generator** of ρ . Any (SCR) cone can be expressed as a Minkowski sum $\sigma = \text{Cone}(G) = \rho_1 + \dots + \rho_r$ of rational rays, where $G = \{u_{\rho_1}, \dots, u_{\rho_r}\}$. Say $R = \mathbb{F}[\sigma^\vee \cap \mathbb{Z}^n]$; we consider R with its \mathbb{Z}^n -grading given by $\deg(\chi^{m_i}) = m_i \in \mathbb{Z}^n$. Setting $\tau^\perp = \{w \in \mathbb{R}^n : \langle w, v \rangle = 0 \text{ for all } v \in \tau\}$, there is a natural graded surjection of \mathbb{Z}^n -graded affine domains over \mathbb{F} :

$$\begin{aligned} \phi_\tau : R = \mathbb{F}[\sigma^\vee \cap \mathbb{Z}^n] &\twoheadrightarrow \mathbb{F}[\sigma^\vee \cap \tau^\perp \cap \mathbb{Z}^n], & \phi_\tau(\chi^m) &= \begin{cases} \chi^m & \text{if } \langle m, v \rangle = 0 \text{ for all } v \in \tau \\ 0 & \text{if } \langle m, v \rangle > 0 \text{ for some } v \in \tau \end{cases} \\ & & &= \begin{cases} \chi^m & \text{if } \langle m, v \rangle = 0 \text{ for all } v \in T \\ 0 & \text{if } \langle m, v \rangle > 0 \text{ for some } v \in T. \end{cases} \end{aligned}$$

Suppose $S_\sigma = \mathbb{Z}_{\geq 0}[m_1, \dots, m_r]$ so that the characters χ^{m_i} generate $R = \mathbb{F}[\chi^{m_1}, \dots, \chi^{m_r}]$ as an \mathbb{F} -algebra. The kernel of ϕ_τ is a graded monomial prime ideal called the **prime ideal of τ** :

$$P_\tau := \ker(\phi_\tau) = (\{\chi^m : \langle m, v \rangle > 0 \text{ for some } v \in T\})R = (\{\chi^{m_i} : \langle m_i, v \rangle > 0 \text{ for some } v \in T\})R.$$

Its height (or codimension) equals the dimension of τ ; in particular, the graded primes in R of height 1 correspond to the rational rays of σ . Conversely, any monomial prime of R corresponds to a face of σ . Moreover, when $\tau = \text{Cone}(T) = \rho_1 + \dots + \rho_\ell$ as a Minkowski sum of rays,

$$P_\tau = \sum_{j=1}^{\ell} P_{\rho_j} \tag{3.0.4}$$

as a sum of ideals. Indeed, setting $T = \{u_{\rho_j} : 1 \leq j \leq \ell\}$, any $v \in \tau$ satisfies

$$v = \sum_{j=1}^{\ell} a_j u_{\rho_j}, \text{ for some } a_1, \dots, a_\ell \in \mathbb{R}_{\geq 0}.$$

Recall that $w \in \sigma^\vee$ if and only if $\langle w, v \rangle \geq 0$ for all $v \in \sigma$. In this case, for $v \in \tau$ as above

$$0 \leq \langle w, v \rangle = \sum_{j=1}^{\ell} a_j \langle w, u_{\rho_j} \rangle, \text{ for some } a_1, \dots, a_\ell \in \mathbb{R}_{\geq 0},$$

and so $\langle w, v \rangle$ is positive if and only if $\langle w, u_{\rho_j} \rangle > 0$ for some $1 \leq j \leq \ell$. We infer from this that the monomial ideals P_τ and $\sum_{j=1}^{\ell} P_{\rho_j}$ have a generating set in common, and hence are equal.

3.0.2. *Finite Tensor Products of Normal Toric Rings.* Fix two (SCR) cones $\sigma' = \text{Cone}(S') \subset \mathbb{R}^m$ and $\sigma'' = \text{Cone}(S'') \subset \mathbb{R}^n$, where S', S'' consist of the ray generators. In terms of ray generators, their **product** is the (SCR) cone generated as

$$\sigma = \sigma' \times \sigma'' = (\sigma' \times \{0\}) + (\{0\} \times \sigma'') = \text{Cone}[(S' \times \{0\}) \cup (\{0\} \times S'')] \subseteq \mathbb{R}^{m+n}.$$

Note that $\sigma^\vee = (\sigma' \times \{0\})^\vee \cap (\{0\} \times \sigma'')^\vee$, since a cone's dual is the intersection of closed half-spaces determined by its ray generators. Suppose $R' = \mathbb{F}[(\sigma')^\vee \cap \mathbb{Z}^m]$, $R'' = \mathbb{F}[(\sigma'')^\vee \cap \mathbb{Z}^n]$. Then the normal toric ring

$$R = \mathbb{F}[\sigma^\vee \cap \mathbb{Z}^{m+n}] \cong R' \otimes_{\mathbb{F}} R'',$$

and every monomial prime Q in R decomposes as $Q = P'R + P''R$ where P' is a monomial prime in R' and P'' is a monomial prime in R'' . Any monomial prime in R corresponds bijectively with a face

of σ . The faces of σ are of the form $\tau' \times \tau''$ where $\tau' = \text{Cone}(T')$ is a face of σ' and $\tau'' = \text{Cone}(T'')$ is a face of σ'' ; here T', T'' consist of the ray generators. Two handy properties of monomial primes in R are: **(1)** $Q_{\tau' \times \tau''} = Q_{\tau' \times \{0\}} + Q_{\{0\} \times \tau''}$ and **(2)** $Q_{\tau' \times \{0\}} = P_{\tau'} R$, $Q_{\{0\} \times \tau''} = P_{\tau''} R$.

We adduce the **Minkowski sum-ideal sum decomposition** (3.0.4) for monomial primes in normal toric rings proven above in verifying both properties: **(1)** is immediate from noticing $\tau' \times \tau'' = (\tau' \times \{0\}) + (\{0\} \times \tau'')$ as a Minkowski sum of faces; for **(2)**, (3.0.4) allows us to reduce verification to the case where τ' and τ'' are rays. We'll do so explicitly for $Q_{\rho' \times \{0\}} = P_{\rho'} R$ where ρ' is a ray of σ' . We will use χ^a, ϕ^b, ψ^c notation for characters in R, R', R'' respectively. We express an arbitrary

$w = (w_1, w_2) \in S_\sigma = \mathbb{Z}^{m+n} \cap \sigma^\vee = [\mathbb{Z}^{m+n} \cap (\sigma' \times \{0\})^\vee] \cap [\mathbb{Z}^{m+n} \cap (\{0\} \times \sigma'')^\vee] = S_{\sigma' \times \{0\}} \cap S_{\{0\} \times \sigma''}$, where $w_1 \in S_{\sigma'} \subseteq \mathbb{Z}^m$ and $w_2 \in S_{\sigma''} \subseteq \mathbb{Z}^n$. Given any $v = (v_1, v_2) \in \sigma$ with $v_1 \in \sigma'$ and $v_2 \in \sigma''$, since $w \in S_\sigma = S_{\sigma' \times \{0\}} \cap S_{\{0\} \times \sigma''}$, all of the dot product terms below are **nonnegative**:

$$\begin{aligned} \langle w, v \rangle &= \langle w, (v_1, 0) \rangle + \langle w, (0, v_2) \rangle = \langle (w_1, 0), (v_1, 0) \rangle + \langle (0, w_2), (0, v_2) \rangle = \langle w_1, v_1 \rangle + \langle w_2, v_2 \rangle \\ \langle (w_1, 0), v \rangle &= \langle w_1, v_1 \rangle \geq 0, \quad \langle (0, w_2), v \rangle = \langle w_2, v_2 \rangle \geq 0. \end{aligned}$$

In particular, since $v \in \sigma$ was arbitrary both $(w_1, 0), (0, w_2) \in S_\sigma$. Thus for $w \in S_\sigma$ as above, the characters $\chi^w, \chi^{(w_1, 0)} = \phi^{w_1}, \chi^{(0, w_2)} = \psi^{w_2} \in R$.

Now suppose $\chi^w = \chi^{(w_1, 0)} \chi^{(0, w_2)} = \phi^{w_1} \psi^{w_2} \in Q_{\rho' \times \{0\}}$, i.e., $\langle w, v \rangle > 0$ for some vector $v = (v_1, v_2) \in \rho' \times \{0\}$. Since $v_2 = 0$ here, equivalently $\langle w, v \rangle = \langle w_1, v_1 \rangle > 0$ for some $v_1 \in \rho'$, i.e., the character $\chi^{(w_1, 0)} = \phi^{w_1} \in P_{\rho'} R$, and since $\chi^{(0, w_2)} = \psi^{w_2} \in R$, $\chi^w = \phi^{w_1} \psi^{w_2} \in P_{\rho'} R$ as well. Thus $Q_{\rho' \times \{0\}} \subseteq P_{\rho'} R$. For the other inclusion: the characters $\chi^{(w_1, 0)} = \phi^{w_1}$ as above generate $P_{\rho'} R$, and each generator lies in $Q_{\rho' \times \{0\}}$ since we noted that $\chi^w \in Q_{\rho' \times \{0\}}$ if and only if $\chi^{(w_1, 0)} = \phi^{w_1} \in P_{\rho'} R$.

Our arguments can be adapted to any finite tensor product $(\otimes_{\mathbb{F}})$ of normal toric rings over \mathbb{F} . In particular, property **(1)** could be stated in a form involving a sum of n monomial primes if we are tensoring n normal toric rings. We summarize this discussion with a

Lemma 3.1. *For $n \geq 2$, let R_1, \dots, R_n be normal toric rings over a field \mathbb{F} . Consider the normal toric ring $R \cong R_1 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n$. Every monomial prime ideal Q in R can be expressed as a sum $Q = \sum_{i=1}^n (P_i R)$ of extended ideals, where each ideal $P_i \subseteq R_i$ is a monomial prime.*

Theorem (1.1) is an immediate corollary, indeed a uniform bound analogue, of the following

Theorem 3.1. *For $n \geq 2$, let R_1, \dots, R_n be normal toric rings over a field \mathbb{F} , $P_i \subseteq R_i$ monomial primes with $1 \leq i \leq n$. For each $1 \leq i \leq n$, suppose there is an integer $D_i > 0$ such that $P_i^{(D_i(r-1)+1)} \subseteq P_i^r$ for all $r > 0$. Set $D = \max\{D_1, \dots, D_n\}$. Then $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$, where the monomial prime $Q = \sum_{i=1}^n (P_i R) \subseteq R \cong R_1 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n$.*

We will state two preliminary lemmas: in both statements S is a **nonzero Noetherian ring**.

Lemma 3.2. *For any prime ideal P in S , and $N \in \mathbb{Z}_{\geq 0}$,*

$$P^{(N)} = P^N :_S (s)^\infty = \bigcup_{j \geq 0} (P^N :_S (s^j)) = P^N :_S (s^T)$$

for all $T \gg 0$ and any $s \notin P$ belonging to all embedded primes of P^N .

Lemma 3.3. *Given any proper ideal I in S , and $E \in \mathbb{Z}_{\geq 0}$,*

$$(1) I^{(N)} \subseteq I^{\lceil N/E \rceil} \text{ for all } N \geq 0 \iff (2) I^{(E(r-1)+1)} \subseteq I^r \text{ for all } r > 0.$$

Proof. The case $N = 0$ is trivial (the unit ideal is contained in itself), so we show equivalence when $N > 0$. Given $r > 0$, setting $N = E(r - 1) + 1$ in (1) gives (2). That (2) implies (1) follows from noticing that for any two positive integers N, r , we have $r = \lceil N/E \rceil$ if and only if $N = E(r - 1) + j$ for some $1 \leq j \leq E$, and $I^{(m)} \subseteq I^{(n)}$ when $m \geq n$. \square

Finally, we adapt Proposition (2.1) to a specialized form suited to the proof. The backdrop will be as follows. Fix a field \mathbb{F} . For $n \geq 2$, fix nonzero \mathbb{F} -algebras R_1, \dots, R_n . Since $R_2 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n \neq 0$ is free and hence faithfully flat over \mathbb{F} , the tensor product $R = R_1 \otimes_{\mathbb{F}} R_2 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n$ is faithfully flat over R_1 ; indeed, R is faithfully flat over each R_i by permuting the tensor factor under consideration (Cf., Exercise 9.11 in Altman-Kleiman [2]). Thus we can view the factors R_i as subrings of R .

Proposition 3.4. *Given the rings R_i and R as above, suppose that R and each factor R_i is Noetherian. Then for each $1 \leq i \leq n$, we have $I^{(N)}R = (IR)^{(N)}$ for all integers $N \geq 0$ where*

$$I \in \mathcal{I}(R_i) = \{\text{proper ideals } I \subseteq R_i : \text{Ass}_R(R/IR) = \{PR : P \in \text{Ass}_{R_i}(R_i/I)\}\}.$$

Moreover, given a pair $(N, r) \in (\mathbb{Z}_{\geq 0})^2$, $I^{(N)} \subseteq I^r$ if and only if $(IR)^{(N)} \subseteq (IR)^r$.

Proof of Theorem (3.1). For each $1 \leq i \leq n$, let $x_{i,1}, \dots, x_{i,t_i}$ be the monomial algebra generators of R_i over \mathbb{F} , so that per the isomorphism $R \cong R_1 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n$, we identify

$$R = \mathbb{F}[\{x_{i,1}, \dots, x_{i,t_i} : 1 \leq i \leq n\}]$$

as an \mathbb{F} -algebra. We define $S(N) := \{(A_1, \dots, A_n) \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=1}^n A_i = N\}$ for each $N \geq 0$, so

$$Q^N = \left(\sum_{i=1}^n (P_i R) \right)^N = \sum_{(A_1, \dots, A_n) \in S(N)} \prod_{i=1}^n (P_i R)^{A_i}.$$

We will in fact show that the monomial ideal

$$Q^{(N)} \subseteq \sum_{(A_1, \dots, A_n) \in S(N)} \prod_{i=1}^n (P_i R)^{(A_i)}. \quad (3.0.5)$$

Take an arbitrary monomial $g = \prod_{i=1}^n m_i \in R$ where m_i is a monomial in the $x_{i,\ell}$ for $1 \leq i \leq n$ and $1 \leq \ell \leq t_i$. Re-indexing if necessary, we may assume that $P_i R = (x_{i,1}, \dots, x_{i,s_i})R$ for $1 \leq i \leq n$ where $1 \leq s_i \leq t_i$. Define a “complement” monomial $\mathcal{M} = \prod_{i=1}^n (x_{i,s_i+1} \dots x_{i,t_i})$ consisting of all algebra generators **not** among the generators of the $P_i R$. Any embedded prime of a power of Q is graded (read, monomial), so that $Q^{(N)} = Q^N :_R (\mathcal{M})^\infty$ as a saturation per Lemma (3.2). If $g \in Q^{(N)}$, then for all $T \gg 0$, the monomial

$$g\mathcal{M}^T \in Q^N = \sum_{(A_1, \dots, A_n) \in S(N)} \prod_{i=1}^n (P_i R)^{A_i},$$

whence for some $(A_1, \dots, A_n) \in S(N)$ and each $1 \leq j \leq n$ we have

$$g\mathcal{M}^T = \prod_{i=1}^n m_i (x_{i,s_i+1} \dots x_{i,t_i})^T = m_j \prod_{i=1}^n m_i^{1-\delta_{ij}} (x_{i,s_i+1} \dots x_{i,t_i})^T \in \prod_{j=1}^n (P_j R)^{A_j},$$

where δ_{ij} is the Kronecker delta. We can express $(P_j R)^{(A_j)} = P_j R^{A_j} :_R (\mathcal{N})^\infty$ where \mathcal{N} is any “complement” monomial built from powers of all algebra generators **not** among the generators of $P_j R$. Therefore, setting $\mathcal{N} = \mathcal{N}(j) = \prod_{i=1}^n m_i^{1-\delta_{ij}} (x_{i,s_i+1} \dots x_{i,t_i})^T$, we see $m_j \in (P_j R)^{(A_j)}$ for each $1 \leq j \leq n$. Thus $g = \prod_{j=1}^n m_j \in \prod_{j=1}^n (P_j R)^{(A_j)}$. Since $g \in Q^{(N)}$ was arbitrary, (3.0.5) is immediate.

Finally, we show that $Q^{(N)} \subseteq Q^{\lceil N/D \rceil}$ for all $N \geq 0$ where $D = \max\{D_1, \dots, D_n\}$. Using (3.0.5):

$$Q^{(N)} \stackrel{(3.0.5)}{\subseteq} \sum_{(A_1, \dots, A_n) \in S(N)} \prod_{i=1}^n (P_i R)^{(A_i)} \stackrel{(1)}{\subseteq} \sum_{(A_1, \dots, A_n) \in S(N)} \prod_{i=1}^n (P_i R)^{\lceil A_i/D_i \rceil} \stackrel{(2)}{\subseteq} Q^{\lceil N/D \rceil}.$$

(1) follows from our key hypothesis: for all $1 \leq i \leq n$, $P_i^{(D_i(r-1)+1)} \subseteq P_i^r$ for all $r > 0$, so $(P_i R)^{(D_i(r-1)+1)} \subseteq (P_i R)^r$ for all $r > 0$ per Proposition (3.4) since $P_i \in \mathcal{I}(R_i)$ per the earlier discussion leading up to Lemma (3.1). Thus per Lemma (3.3), $(P_i R)^{(A_i)} \subseteq (P_i R)^{\lceil A_i/D_i \rceil}$ for all $A_i \geq 0$ and all $1 \leq i \leq n$. For (2) we show that for each $(A_1, \dots, A_n) \in S(N)$, we have $\prod_{i=1}^n (P_i R)^{\lceil A_i/D_i \rceil} \subseteq Q^{\lceil N/D \rceil}$: indeed, $\lceil A_i/D_i \rceil \geq \lceil A_i/D \rceil$ for each $1 \leq i \leq n$, and the integer $\sum_{i=1}^n \lceil A_i/D \rceil \geq \lceil (\sum_{i=1}^n A_i)/D \rceil = \lceil N/D \rceil$ for each $(A_1, \dots, A_n) \in S(N)$. To finish the proof, since $Q^{(N)} \subseteq Q^{\lceil N/D \rceil}$ for all $N \geq 0$, we simply invoke Lemma (3.3) again. \square

4. PROVING THEOREM 1.2 IN A REFINED FORM

Theorem (1.2) is easy if $n = 1$ or $D = 1$: all rings in sight are polynomial rings and monomial primes are complete intersections. Thus for the remainder of this section, **we will assume that $n \geq 2$ and $D \geq 2$** . We will give presentations of our rings as subrings of the domain of Laurent polynomials $L = \mathbb{F}[s_1^{\pm 1}, \dots, s_{n-1}^{\pm 1}, u^{\pm 1}]$ in n indeterminates over the field \mathbb{F} . The proof will proceed in cases, starting with the ring

$$H_D = \frac{\mathbb{F}[x_1, \dots, x_n, z]}{(z^D - x_1 \cdots x_n)}.$$

4.0.3. *The Hypersurface Case:* We first observe that H_D is a toric ring, up to isomorphism:

Lemma 4.1. *Consider the full-dimensional strongly convex, rational polyhedral cone $\sigma_D^{(n)} \subseteq \mathbb{R}^n$ whose ray generators are $De_i + e_n$ for $1 \leq i < n$ and e_n in terms of the standard basis vectors.*

- (1) *The Hilbert basis of the semigroup $(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n$ consists of $n+1$ vectors: the n standard basis vectors e_1, \dots, e_n , together with the vector $(-1, \dots, -1, D) \in \mathbb{Z}^n$.*
- (2) *The toric ring $\mathbb{F}[(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n] \cong \frac{\mathbb{F}[x_1, \dots, x_n, z]}{(z^D - x_1 \cdots x_n)} = H_D$.*

Proof. The reader can use the hilbertBasis algorithm implemented in the Polyhedra package in Macaulay2 ([9]) to check (1). For (2), recall from Section (3) that if $\sigma \subseteq \mathbb{R}^n$ is a (SC-R) cone, then to each $m = (m_1, \dots, m_{n-1}, m_n) \in \sigma^\vee \cap \mathbb{Z}^n$ we assign a Laurent monomial $\chi^m = s_1^{m_1} \cdots s_{n-1}^{m_{n-1}} u^{m_n}$ in the semigroup ring $\mathbb{F}[\sigma^\vee \cap \mathbb{Z}^n]$. Given (1), we have

$$\mathbb{F}[(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n] = \mathbb{F}\left[s_1, \dots, s_{n-1}, \frac{u^D}{(s_1 \cdots s_{n-1})}, u\right] \subseteq \mathbb{F}[s_1^{\pm 1}, \dots, s_{n-1}^{\pm 1}, u^{\pm 1}].$$

Given a polynomial ring $R = \mathbb{F}[x_1, \dots, x_{n-1}, x_n, z]$ in $n+1$ variables, consider the surjective algebra map $\phi: R = \mathbb{F}[x_1, \dots, x_{n-1}, x_n, z] \twoheadrightarrow \mathbb{F}[(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n]$ under which $x_i \mapsto s_i$ for each $1 \leq i \leq n-1$, $x_n \mapsto \frac{u^D}{(s_1 \cdots s_{n-1})}$, and $z \mapsto u$. Since $\dim(R) = \dim(\mathbb{F}[(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n]) + 1$, we conclude that the kernel of ϕ is a height one prime in the UFD R , and hence is principal. Now $F = z^D - x_1 \cdots x_n \in R$ is irreducible and belongs to the kernel of ϕ , so $\ker \phi = (F)$, and the isomorphism claim follows. \square

We now deduce the following refinement of Theorem (1.2) for H_D :

Theorem 4.1. Take the ring $H_D = \mathbb{F}[x_1, \dots, x_n, z]/(z^D - x_1 \cdots x_n)$, and P one of the monomial prime ideals of H_D (i.e., \mathbb{Z}^n -graded /torus-invariant); assume P is nonzero and nonmaximal. When $D \leq \text{ht}(P)$ (the **height** of P), $P^{(N)} = P^N$ for all $N > 0$. If $D \geq \text{ht}(P)$ and $N \equiv 1 \pmod{D}$, then

$$P^{(N)} \subseteq P^{\text{ht}(P)(\frac{N-1}{D})+1}.$$

In particular, $P^{(Dr)} \subseteq P^{(D(r-1)+1)} \subseteq P^{\text{ht}(P)(r-1)+1} \subseteq P^r$ for all $r > 0$.

Proof. To start, define the height j prime ideal $P_j = (z, x_1, \dots, x_j)H_D$ for all $1 \leq j \leq n-1$; under the correspondence in subsection (3.0.1), $P_j = P_\tau$ for the j -dimensional face τ of $\sigma_D^{(n)}$ generated by $De_i + e_n$ for $1 \leq i \leq j$. As a saturation, $P_j^{(N)} = P_j^N :_{H_D} (\prod_{i=j+1}^n x_i)^\infty$. Since $P_j^{(N)}$ is monomial, in chasing down inclusions below it suffices to discern which monomial classes

$$g = (z^\ell x_1^{a_1} \cdots x_j^{a_j})(x_{j+1}^{a_{j+1}} \cdots x_n^{a_n}) \in H_D$$

multiply a power of $m = \prod_{i=j+1}^n x_i$ into P_j^N . For g as above, by definition $g \in P_j^{(N)}$ if and only if for all $M \gg 0$,

$$\begin{aligned} P_j^N \ni m^M g &= z^\ell \left(\prod_{i=j+1}^n x_i^{a_i+M} \right) \left(\prod_{i=1}^j x_i^{a_i} \right) = z^\ell \left(\prod_{i=1}^n x_i \right)^{M'} \left(\prod_{i=j+1}^n x_i^{a_i+M-M'} \right) \left(\prod_{i=1}^j x_i^{a_i-M'} \right) \\ &= \left(z^{D \cdot M' + \ell} \prod_{i=1}^j x_i^{a_i-M'} \right) \left(\prod_{i=j+1}^n x_i^{a_i+M-M'} \right) \end{aligned}$$

where $M' = M'(M) := \min(a_1, \dots, a_j, a_{j+1} + M, \dots, a_n + M) = \min(a_1, \dots, a_j)$ for all $M \gg 0$.

We conclude that $z^{D \cdot M' + \ell} \left(\prod_{i=1}^j x_i^{a_i-M'} \right) \in P_j^N$, and infer the inequality

$$(D-j)M' + \left(\sum_{i=1}^j a_i \right) + \ell \geq N. \quad (4.0.6)$$

Before proceeding, notice that since $M' \geq 0$, when $D \leq j$ so that the number $(D-j)M'$ is nonpositive, (4.0.6) implies that $\left(\sum_{i=1}^j a_i \right) + \ell \geq N$, so $(z^\ell x_1^{a_1} \cdots x_j^{a_j}) \in P_j^N$ and hence $g \in P_j^N$ already. Thus $P_j^{(N)} = P_j^N$ for all $N > 0$ when $D \leq j$, since both are generated by monomial classes. Thus in the remainder of the proof **we will assume that** $D \geq j = \text{ht}(P_j)$ (i.e., $D-j \geq 0$). In this case, assuming $N \equiv 1 \pmod{D}$, we now show that $P_j^{(N)} \subseteq P_j^{1+j(\frac{N-1}{D})}$. Fix a monomial

$$g = \left(z^\ell \prod_{i=1}^j x_i^{a_i} \right) \left(\prod_{i=j+1}^n x_i^{a_i} \right) \in P_j^{(N)}$$

and $M' = \min(a_1, \dots, a_j)$ exactly as before and set $E := \ell + \sum_{i=1}^j a_i$. The more involved case for us is when **(**)** $M' \leq (N-1)/D$: otherwise $E \geq a_1 + \cdots + a_j \geq jM' \geq j(N-1)/D + 1$, whence one easily infers that $g \in P_j^{j(\frac{N-1}{D})+1}$. Assuming **(**)**, we now show that $E \geq j(\frac{N-1}{D}) + 1$. Suppose to the contrary that $E \leq j(\frac{N-1}{D})$. Since $g \in P_j^{(N)}$, inequality (4.0.6) above says

$$(D-j)M' + E = (D-j)M' + \left(\sum_{i=1}^j a_i \right) + \ell \geq N \implies E \geq N - (D-j)M'.$$

Then since $N - 1 - DM' \geq 0$ by (**), and $(D - j) \geq 0$, we see that

$$\begin{aligned}
j(N - 1) &= Dj \left(\frac{N - 1}{D} \right) \geq DE \geq DN - D(D - j)M' \\
&= D(N - 1) + D - D(D - j)M' \\
&= j(N - 1) + D + (D - j)(N - 1 - DM') \\
&\geq j(N - 1) + D + (D - j)(0) \\
&= j(N - 1) + D
\end{aligned}$$

a contradiction. Thus $E \geq j \left(\frac{N-1}{D} \right) + 1$, so $g \in P_j^{1+j \left(\frac{N-1}{D} \right)}$. In particular, when $N = D(r-1)+1$, we have $P_j^{(D(r-1)+1)} \subseteq P_j^{1+j(r-1)}$. Finally, applying coordinate changes according to every permutation of $x_{[n]} := \{x_1, \dots, x_n\}$, any (nonzero, nonmaximal) monomial prime ideal in H_D can be obtained from the P_j running through all $1 \leq j \leq n-1$, along with obtaining the desired containments. \square

4.0.4. *The Veronese Case:* Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ denote the set of **nonnegative integers**. To start,

Lemma 4.2. *Consider the full-dimensional strongly convex, rational polyhedral cone $\eta_D^{(n)} \subseteq \mathbb{R}^n$ whose ray generators are e_i for $1 \leq i < n$ and $(-1, -1, \dots, -1, D)$ in terms of the standard basis vectors.*

(1) *The Hilbert basis of the semigroup $(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n$ is the set of vectors*

$$\left\{ (a_1, \dots, a_{n-1}, 1) \in \mathbb{N}^n : 0 \leq \sum_{i=1}^{n-1} a_i \leq D \right\}.$$

(2) *The toric ring $\mathbb{F}[(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n] \cong V_D$, the D -th Veronese subring of the polynomial ring $\mathbb{F}[s_1, \dots, s_{n-1}, u]$ in the n indeterminates s_1, \dots, s_{n-1}, u .*

Proof. The reader can use the hilbertBasis algorithm implemented in the Polyhedra package in Macaulay2 ([9]) to check (1). Given (1), as an algebra over \mathbb{F} , we have

$$\begin{aligned}
\mathbb{F}[(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n] &= \mathbb{F} \left[s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u : \text{each } a_i \geq 0, 0 \leq \sum_{i=1}^{n-1} a_i \leq D \right] \\
&\cong \frac{\mathbb{F}[x_{(a_1, \dots, a_{n-1})} : \text{each } a_i \geq 0, 0 \leq \sum_{i=1}^{n-1} a_i \leq D]}{(x_e x_f - x_g x_h : e + f = g + h \in \mathbb{N}^{n-1})}.
\end{aligned}$$

Within the polynomial ring $\mathbb{F}[s_1, \dots, s_{n-1}, u]$, applying the correspondence

$$s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u \longleftrightarrow s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u^{D-a_1-\dots-a_{n-1}}$$

takes the generators in the presentation of $\mathbb{F}[(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n]$ and recovers the usual presentation of V_D in terms of degree D monomials in n variables. Therefore, (2) holds: $\mathbb{F}[(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n] \cong V_D$. \square

We use the toric presentation of V_D to deduce the following refinement of Theorem (1.2) for V_D :

Theorem 4.3. *Over an arbitrary field \mathbb{F} , take the D -th Veronese subring $V_D \subseteq \mathbb{F}[x_1, \dots, x_n]$ and P one of the monomial prime ideals of V_D . When P is nonzero and nonmaximal,*

$$P^{(N)} \subseteq P^r \iff r \leq \lceil N/D \rceil;$$

in particular, $P^{(Dr)} \subseteq P^{(D(r-1)+1)} \subseteq P^r$ for all $r > 0$ and the right-hand containment is sharp.

Proof. For all $1 \leq j \leq n-1$, define height one primes

$$P_j = P_{e_j} = \left(s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u : a_j > 0, \text{ and } 1 \leq \sum_{b=1}^{n-1} a_b \leq D \right) V_D.$$

Then by the **Minkowski sum-ideal sum decomposition** (3.0.4) $P_{j_1 < \dots < j_k} := P_{j_1} + \dots + P_{j_k}$ is a prime of height $1 \leq k \leq n-1$ for each size- k subset $j_1 < \dots < j_k$ of $[n-1] = \{1, \dots, n-1\}$. In particular, we focus on $P_{1 < \dots < k} = (s^{\bar{a}} u : \bar{a} \in T_k) V_D$, where

$$T_k := \left\{ \bar{a} = (a_1, \dots, a_{n-1}) \in \mathbb{N}^{n-1} : 1 \leq \sum_{b=1}^k a_b \leq \sum_{b=1}^{n-1} a_b \leq D \right\}.$$

Any monomial g in $P_{1 < \dots < k}^{(N)} \subseteq P_{1 < \dots < k} \subseteq P_{1 < \dots < n-1}$ belongs to $P_{1 < \dots < k}$ and so decomposes (for some $B \geq 0$) as

$$g = u^B \prod_{\bar{a} \in T_{n-1}} (s^{\bar{a}} u)^{i_{\bar{a}}} = \prod_{\bar{a} \in T_k} (s^{\bar{a}} u)^{i_{\bar{a}}} \left(u^B \prod_{\bar{a} \in T_{n-1} - T_k} (s^{\bar{a}} u)^{i_{\bar{a}}} \right) \in P_{1 < \dots < k}^{\sum_{\bar{a} \in T_k} i_{\bar{a}}}.$$

Note that this factorization of g into two monomial pieces (T_k versus $T_{n-1} - T_k$) is unique up to applying the Veronese relations $s^{\bar{e}} u \cdot s^{\bar{f}} u = s^{\bar{g}} u \cdot s^{\bar{h}} u$ ($\bar{e} + \bar{f} = \bar{g} + \bar{h}$). Setting the monomial $m := u \cdot \prod_{\bar{a} \in T_{n-1} - T_k} s^{\bar{a}} u \in V_D$ to be the **product** of the monomials $s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u$ with $a_j = 0$ for all $1 \leq j \leq k$ ($\leq n-1$), we have $P_{1 < \dots < k}^{(N)} = P_{1 < \dots < k}^N : V_D (m)^\infty$, and the monomial g is in $P_{1 < \dots < k}^{(N)}$ precisely when for all $M \gg 0$,

$$g \cdot m^M = \left(u^{B+M} \prod_{\bar{a} \in T_k} (s^{\bar{a}} u)^{i_{\bar{a}}} \right) \prod_{\bar{a} \in T_{n-1} - T_k} (s^{\bar{a}} u)^{i_{\bar{a}} + M} \in P_{1 < \dots < k}^N.$$

In particular, the monomial in parentheses is in $P_{1 < \dots < k}^N$ so it is a multiple of some N -fold product of generators of $P_{1 < \dots < k} = (s^{\bar{a}} u : \bar{a} \in T_k) V_D$. Thus we infer that two inequalities must hold, signifying we have enough u 's and s_j 's ($1 \leq j \leq k$) at our disposal, respectively, to feasibly form such a N -fold product. These inequalities are **(1)** $\sum_{\bar{a} \in T_k} i_{\bar{a}} + B + M \geq N$, and **(2)** the sum

$$\sum_{\bar{a} \in T_k} i_{\bar{a}} (a_1 + \dots + a_k) = \sum_{j=1}^D \ell_j \cdot j \geq N,$$

where $\ell_j := \sum_{\bar{a} \in T_{k,j}} i_{\bar{a}}$, defining $T_{k,j} := \{\bar{a} \in T_k : \text{the partition } a_1 + \dots + a_k = j\}$. Indeed,

$$N \leq \sum_{j=1}^D \ell_j \cdot j \leq D \left(\sum_{j=1}^D \ell_j \right) \implies \sum_{j=1}^D \ell_j \geq \lceil N/D \rceil,$$

so **(2)** implies that **(3)** $\sum_{\bar{a} \in T_k} i_{\bar{a}} = \sum_{j=1}^D \ell_j \geq \lceil N/D \rceil$.¹ For any monomial $g \in P_{1 < \dots < k}^{(N)}$, **(3)** implies that $g \in P_{1 < \dots < k}^{\lceil N/D \rceil}$. Thus $P_{1 < \dots < k}^{(N)} \subseteq P_{1 < \dots < k}^{\lceil N/D \rceil}$ for all $N > 0$.

Additionally if we consider R with its **standard N-grading**, then the minimal degree of a monomial (e.g., a monomial generator) in $P_{1 < \dots < k}^r$ is r . Noticing that for $1 \leq j \leq k$, the degree

¹Together, inequalities **(1)** and **(3)** are equivalent to

$$\sum_{\bar{a} \in T_k} i_{\bar{a}} = \sum_{j=1}^D \ell_j \geq \max\{\lceil N/D \rceil, N - (B + M)\} \equiv \lceil N/D \rceil \text{ for all } M \geq N.$$

$\lceil N/D \rceil$ monomial $(s_j^D u)^{\lceil N/D \rceil} \in P_{1 < \dots < k}^N : (u^{(N+1) - \lceil N/D \rceil}) \subseteq P_{1 < \dots < k}^N : (m^{(N+1) - \lceil N/D \rceil}) \subseteq P_{1 < \dots < k}^{(N)}$, we obtain the only-if part of: for each $1 \leq k \leq n$, $P_{1 < \dots < k}^{(N)} \subseteq P_{1 < \dots < k}^r$ if and only if $r \leq \lceil N/D \rceil$.

Setting $N = Dr - (D - 1) = D(r - 1) + 1$, we have $\lceil N/D \rceil = (r - 1) + 1/D = r$, so that $P_{1 < \dots < k}^{(Dr - (D - 1))} \subseteq P_{1 < \dots < k}^r$ for all $r > 0$ and this containment is sharp.

In review, our argument **does not** depend crucially on which size- k index subset $j_1 < \dots < j_k$ of $[n] = \{1, 2, \dots, n\}$ we worked with; going with $1 < 2 < \dots < k$ merely simplifies notation. In other words, in applying suitable permutations of the algebra generators for V_D , one obtains the above characterization of ideal containment for all of the monomial prime ideals in the ring **having one of the P_j as an ideal summand**. To handle monomial primes having the height one prime

$$P_{(-1, \dots, -1, D)} = \left(s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u : 0 \leq \sum_{i=1}^{n-1} a_i \leq D - 1 \right)$$

as a summand, we use the \mathbb{F} -algebra isomorphisms $\phi_j : V_D \rightarrow V_D$ ($1 \leq j \leq n - 1$) under which a monomial algebra generator $g = s_1^{a_1} \cdots s_j^{a_j} \cdots s_{n-1}^{a_{n-1}} u$ with $0 \leq A := \sum_{i=1}^{n-1} a_i \leq D$ is sent to

$$\phi_j(g) = \begin{cases} s_1^{a_1} \cdots s_j^{D-A} \cdots s_{n-1}^{a_{n-1}} u & \text{if } A \leq D - 1 \text{ and } a_j = 0 \\ s_1^{a_1} \cdots s_j^0 \cdots s_{n-1}^{a_{n-1}} u & \text{if } A = D \text{ and } a_j > 0 \\ g & \text{if } A \leq D - 1 \text{ and } a_j > 0 \\ g & \text{if } A = D \text{ and } a_j = 0. \end{cases}$$

We note that $\phi_j^2 = \phi_j \circ \phi_j$ is the identity, and the height one prime $\phi_j(P_{(-1, \dots, -1, D)}) = P_j$: indeed, when $h = s_1^{a_1} \cdots s_j^{a_j} \cdots s_{n-1}^{a_{n-1}} u$ is a generator of P_j , $a_j > 0$; when $A \leq D - 1$, $h = \phi_j(h)$, or else $D - A = 0$, $a_j = D - \left(\sum_{1 \leq i \neq j \leq n-1} a_i \right) > 0$, and $h = \phi_j(g)$ where $g = s_1^{a_1} \cdots s_j^0 \cdots s_{n-1}^{a_{n-1}} u \in P_{(-1, \dots, -1, D)}$. Moreover, we conclude that a (sharp) containment $Q^{(m)} \subset Q^r$ for any monomial prime Q with P_j as a summand translates under ϕ_j to a (sharp) containment $(Q')^{(m)} \subset (Q')^r$ for a monomial prime Q' of the same height as Q , with $P_{(-1, \dots, -1, D)}$ replacing P_j as an ideal summand. Having analyzed ideals with one of the P_j as a summand quite thoroughly, this final observation completes the proof. \square

4.0.5. Connection to Divisor Class Groups. In this closing subsection only, and in keeping with our main sources ([5, 8]), we will assume that \mathbb{F} is algebraically closed. For a fixed (SCR) cone $\sigma \subseteq \mathbb{R}^n$, and without being explicit, the faces τ of σ are in bijection with the following sets:

- {torus-invariant closed subvarieties of U_σ }, and
- {torus-invariant ($= \mathbb{Z}^n$ -graded/monomial) prime ideals of $R = \mathbb{F}[U_\sigma]$ }

via $\tau \mapsto V(\tau) = \mathbb{V}(P_\tau) \mapsto P_\tau$. We note that $\dim(V(\tau)) = \text{codim}(\tau)$, and that the height/codimension of P_τ equals $\text{codim}(V(\tau)) = \dim(\tau)$. Going forward, we let $\Sigma(1)$ denote the collection of **rays** (one-dimensional faces). In particular, each ray $\rho \in \Sigma(1)$ has a generator $u_\rho \in \rho \cap \mathbb{Z}^n$ that is primitive, and yields a torus-invariant prime divisor $D_\rho = V(\rho)$ on $X = U_\sigma$. Per Exercise 4.1.1 of ([5]), we note that $\text{Div}_T(X) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho$ is the free abelian group of **torus-invariant Weil divisors** on X . We recap parts (2) and (3) of Theorem 3.1 in ([20]).

Theorem 4.2. *Take a normal affine toric variety $X = U_\sigma$, $\text{Cl}(X)$ its divisor class group. Then*

- (1) $\text{Cl}(X)$ is finite abelian if and only if σ is simplicial. If so, then Weil divisors on X are \mathbb{Q} -Cartier of index bounded above by the order of $\text{Cl}(X)$.
- (2) If $G = \{u_\rho : \rho \in \Sigma(1)\}$ spans \mathbb{R}^n , then the following sequence of abelian groups is exact

$$0 \rightarrow \mathbb{Z}^n \xrightarrow{\phi} \text{Div}_T(X) \rightarrow \text{Cl}(X) \rightarrow 0,$$

where $\phi(m) = \text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$, and $\langle \cdot, \cdot \rangle$ is dot product. In particular, when $\sigma \subseteq \mathbb{R}^n$ is **full (SCRF)**,

$$\text{Cl}(\mathbb{F}[\sigma^\vee \cap \mathbb{Z}^n]) \cong \frac{\mathbb{Z}[\{[D_\rho] : \rho \in \Sigma(1)\}]}{\langle \sum_{\rho \in \Sigma(1)} \langle e_i, u_\rho \rangle [D_\rho] = 0 : 1 \leq i \leq n \rangle}$$

where the $e_i \in \mathbb{Z}^n$ are the standard basis vectors.

Examples: We revisit the (SCRF) polyhedral cones in the proof of Theorem (1.2), showing that $\text{Cl}(H_D) \cong (\mathbb{Z}/D\mathbb{Z})^{n-1}$ and $\text{Cl}(V_D) \cong \mathbb{Z}/D\mathbb{Z}$. Although these class group facts are well known in certain circles and can be deduced by other means (see e.g., [18]), for completeness of exposition we include some succinct computations using toric divisor theory.

- (1) The cone $\sigma_D^{(n)} \subseteq \mathbb{R}^n$ has ray generators $f_i = De_i + e_n$ for $1 \leq i < n$ and e_n , and

$$\begin{aligned} \text{Cl}(\mathbb{F}[(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n]) &\cong \frac{\mathbb{Z}([D_{f_1}], \dots, [D_{f_{n-1}}], [\mathbf{D}_{e_n}])}{\langle D[D_{f_1}] = 0, \dots, D[D_{f_{n-1}}] = 0, [\mathbf{D}_{e_n}] = -[\mathbf{D}_{f_1}] - \dots - [\mathbf{D}_{f_{n-1}}] \rangle} \\ &\cong \frac{\mathbb{Z}([D_{f_1}], \dots, [D_{f_{n-1}}], -[\mathbf{D}_{f_1}] - \dots - [\mathbf{D}_{f_{n-1}}])}{\langle D[D_{f_1}] = 0, \dots, D[D_{f_{n-1}}] = 0 \rangle} \\ &= \frac{\mathbb{Z}([D_{f_1}], \dots, [D_{f_{n-1}}])}{\langle D[D_{f_1}] = 0, \dots, D[D_{f_{n-1}}] = 0 \rangle} \cong \boxed{(\mathbb{Z}/D\mathbb{Z})^{n-1}}. \end{aligned}$$

- (2) The cone $\eta_D^{(n)} \subseteq \mathbb{R}^n$ has ray generators e_i for $1 \leq i < n$ and $f_n = (-1, -1, \dots, -1, D)$, and

$$\text{Cl}(\mathbb{F}[(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n]) \cong \frac{\mathbb{Z}([D_{e_1}], \dots, [D_{e_{n-1}}], [D_{f_n}])}{\langle [\mathbf{D}_{e_i}] - [\mathbf{D}_{f_n}] = \mathbf{0} \ (1 \leq i < n), D[D_{f_n}] = 0 \rangle} \cong \frac{\mathbb{Z}([D_{f_n}])}{\langle D[D_{f_n}] = 0 \rangle} \cong \boxed{(\mathbb{Z}/D\mathbb{Z})}.$$

5. LINGERING QUESTIONS RELATED TO THEOREM 1.1

To summarize, we have deduced two existence criteria for uniform Harbourne-Huneke bounds. Lemma (1.1) holds for ideals of pure height one in a Noetherian normal domain. And Theorem (1.1) holds for monomial primes in **finite tensor products** of normal toric rings; we deduced Theorem (1.2) to increase the range of examples that can be used as tensor factors. These criteria cover a reasonably **prodigious** class of normal toric rings. We close with a few natural lines for further investigation.

- (1) Does the conclusion of Theorem (1.1) extend to monomial primes in any simplicial toric ring built from a (SCRF) cone? Can we identify a candidate mechanism (e.g., group-theoretic) to help explain and verify these Harbourne-Huneke bounds in height two or higher for a larger class of ideals than monomial primes?
- (2) Given the role of tensor products in our manuscript, do analogues of Theorems (1.1) and (3.1) hold for other graded ring constructions in the toric setting, such as Segre products?

REFERENCES

- [1] S. Akeseh. *Ideal Containments Under Flat Extensions*. arXiv preprint
- [2] A. Altman and S. Kleiman. *A Term of Commutative Algebra*. Worldwide Center of Mathematics LLC, Cambridge, MA, 2014.
- [3] M. Atiyah, I.G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley, Reading, MA, 1969.
- [4] C. Bocci, and B. Harbourne. *Comparing powers and symbolic powers of ideals*. J. Algebraic Geom. **19** (2010), no.3, pp. 399-417. arXiv preprint
- [5] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties* (2011), Graduate Studies in Mathematics 124. American Mathematical Society, Providence, RI.
- [6] M. Dumnicki, T. Szemberg, and H. Tutaj-Gasińska. *Counterexamples to the $I^{(3)} \subseteq I^2$ containment*. J. Algebra 393 (2013) pp.24-29. arXiv preprint
- [7] L. Ein, R. Lazarsfeld, and K. Smith. *Uniform bounds and symbolic powers on smooth varieties*. Invent. Math. **144** (2001), pp. 241-252. arXiv preprint
- [8] W. Fulton. *Introduction to Toric Varieties* (1993), Annals of Math. Studies 131. Princeton University Press, Princeton, NJ.
- [9] D.R. Grayson and M.E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at <http://www.uiuc.edu/Macaulay2/>.
- [10] B. Harbourne and C. Huneke. *Are symbolic powers highly evolved?*. J. Ramanujan Math. Soc. **28A** (2013), pp. 247-266. arXiv preprint
- [11] B. Harbourne and A. Seceleanu. *Containment counterexamples for ideals of various configurations of points in \mathbb{P}^n* . J. Pure Appl. Algebra **219** (2015), no.4, pp. 1062-1072. arXiv preprint
- [12] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Math. **52**, Springer-Verlag, New York, 1977.
- [13] M. Hochster. *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*. Ann. of Math (2) **96** (1972), pp. 318-337.
- [14] M. Hochster. *Math 615 Winter 2007 Lecture 4/6/07*. Online link.
- [15] M. Hochster and C. Huneke. *Comparison of ordinary and symbolic powers of ideals*. Invent. Math. **147** (2002), pp. 349-369. arXiv preprint
- [16] J. Lipman. *Rational singularities, with applications to algebraic surfaces and unique factorization*. Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, pp. 195-279.
- [17] H. Matsumura. *Commutative Ring Theory*. Cambridge Univ. Press, Cambridge, (1989).
- [18] A.K. Singh and S. Spiroff. *Divisor class groups of graded hypersurfaces*. Contemporary Mathematics **448** (2007) 237-243.
- [19] T. Szemberg, J. Szpond. *On the containment problem*. arXiv preprint
- [20] R.M. Walker. *Rational Singularities and Uniform Symbolic Topologies*. arXiv preprint

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI, 48109

E-mail address: `robmarsw@umich.edu`